Math 436 midterm (practice)

Name: ____________________

This exam has 8 questions, for a total of 100 points.

Please answer each question in the space provided. No aids are permitted.

Question 1. (10 pts)

(a) State the definition of a topology on a set X.

(b) Find a family of open subsets of the real line $\mathbb{R}$ whose intersection is not open.

Question 2. (10 pts)

(a) State the definition of compactness.

(b) Is it possible for a discrete space to be compact? Explain.

Question 3. (10 pts)

Let $f\colon X \to \mathbb{R}$ be a continuous function on a topological space X. Suppose U is an open set of X, is $f(U)$ always open in $\mathbb{R}$? Explain.

Question 4. (10 pts)

Suppose X is compact space and $f\colon X \to \mathbb{R}$ is a continuous real valued function on X. If $f(x) > 0$ for all $x \in X$, prove that there exists a number $r > 0$ such that $f(x) > r$ for all $x \in X$.

Question 5. (15 pts)

Let (X, d_X) and (Y, d_Y) be two metric spaces. Suppose $f\colon X \to Y$ is a map such that $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$.

(a) Prove that f is injective.

(b) Prove that f is continuous.

Question 6. (10 pts)

Let X be a discrete topological space with at least two distinct points. Show that X is not connected.

Question 7. (15 pts)

(a) Let A and B be two connected subsets of $\mathbb{R}$. If $A \cap B \neq \emptyset$, show that $A \cap B$ is connected.

(b) Find two connected subsets C and D of $\mathbb{R}^2$ such that $C \cap D \neq \emptyset$ and $C \cap D$ is *not* connected.

Question 8. (20 pts)

Let $f\colon [0,1] \to \mathbb{R}$ be a continuous real-valued function on $[0,1]$. The graph $G(f)$ of f is defined to be the following subset of $\mathbb{R}^2$:

$$\big\{(x, f(x)) \in \mathbb{R}^2 \mid x \in [0,1]\big\}.$$

(a) Show that $G(f)$ is compact.

(b) Show that $G(f)$ is homeomorphic to $[0,1]$.